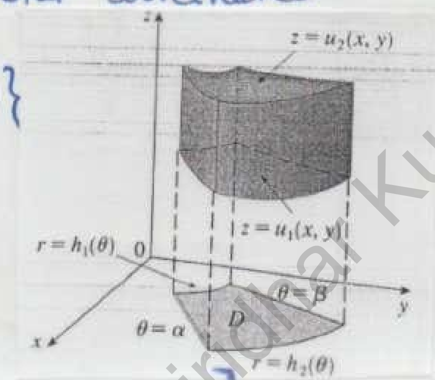
Evaluating Triple Integrals w/ Cylindrical Coordinates:

If E is a region (suppose that it is a type I region) whose projection D onto the xy-plane is conveniently described in polar coordinates:



$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

& D is given in polar coordinates by

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

we know that $\displaystyle\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$

and we know how to evaluate double integrals in polar coordinates, we can rewrite the above equation as:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

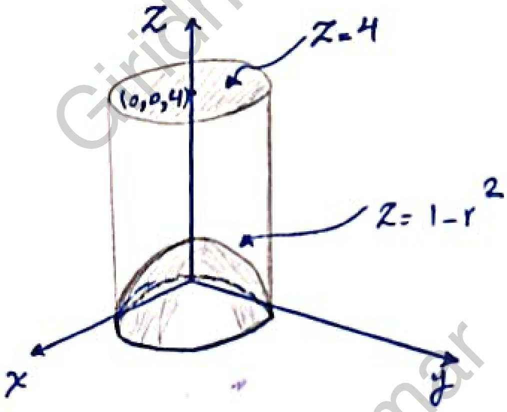
Note: The above equation says that to convert a triple integral from rectangular to cylindrical coordinates, we should write $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using appropriate limits of integration for z, r, and θ and replacing dV by $r\,dz\,dr\,d\theta$.

It is worthwhile to use cylindrical coordinates, especially when $f(x,y,z)$ involves x^2+y^2 expression.

Example: A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution: in cylindrical coordinates, the equation of a cylinder is $r=1 \to$ why? The equation of a paraboloid is $z=1-r^2$ (why? Think about it)

if we draw region E, we will have:



$$E=\{(r,\theta,z) \mid 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}$$

since the density at (x,y,z) is proportional to the distance from z-axis

$$\Rightarrow f(x,y,z) \underset{\text{proportionality sign}}{\propto} \sqrt{x^2+y^2}$$

$$\Rightarrow f(x,y,z) = \underset{\text{Proportionality Constant}}{K}\sqrt{x^2+y^2}$$

$$\Rightarrow m=\iiint_E K\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (kr)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 \left[kr^2\cdot z\right]_{1-r^2}^4 dr\,d\theta=\int_0^{2\pi}\int_0^1 kr^2\left(4-(1-r^2)\right)dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 (3kr^2+kr^4)\,dr\,d\theta=k\int_0^{2\pi}\left[r^3+\frac{r^5}{5}\right]_0^1 d\theta$$

$$=k\int_0^{2\pi}\left(1+\frac{1}{5}\right)d\theta=k\left[\frac{6}{5}\theta\right]_0^{2\pi}=k\cdot\frac{6}{5}\cdot 2\pi=\frac{12k\pi}{5}$$